Probabilities and distributions

Johan A. Elkink

University College Dublin

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1. Probabilities
2. Probability distributions
3. Normal distribution
4. Exercises

Outline

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Probability: definition

Frequentist approach: a probability is the proportion of times a particular event will occur given a long run of repeated observations / experiments.

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Subjective (Bayesian) approach: a probability is the formal quantification of the subjective belief about how likely a certain event will occur when an observation / experiment is performed.

Probability: properties

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- If S contains all possible events, $Pr(S) = 1$
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- If A and C are independent events, $Pr(AC) = Pr(A)Pr(C)$

Probability: example

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- $Pr(\text{head}) = \frac{1}{2}$
- $Pr(\text{tail}) = \frac{1}{2}$
- Sum of all possible events:
  $Pr(\text{head}) + Pr(\text{tail}) = \frac{1}{2} + \frac{1}{2} = 1$
- Probability of head and then tail:
  $Pr(\text{head, then tail}) = Pr(\text{head})Pr(\text{tail}) = \frac{1}{4}$

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For interval and ratio variables, the probability distribution is a continuous function, the "probability density function" (p.d.f.).

Probability distributions: properties

Probability mass function: all probabilities together sum to 1.

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Probability density function: the total area under the graph is 1.

Probability distributions: properties

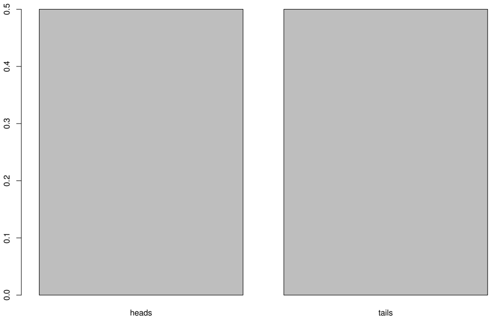

Figure : Example of probability distribution

Probability distributions: properties

For a continuous variables, you can only speak of the probability that the observation will be within a certain range of values, not of the probability of a specific value.

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The probability is represented by the area under the graph over that range.

Probability distributions: properties

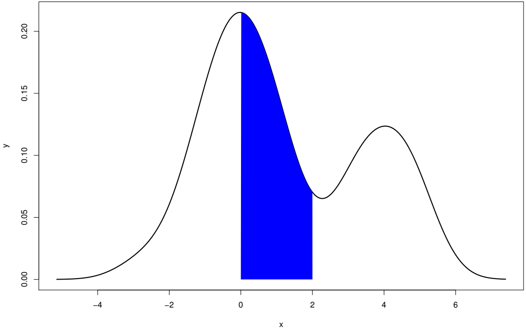

Figure : Example of probability distribution, $Pr(0 < X < 2)$

Probability distributions: properties

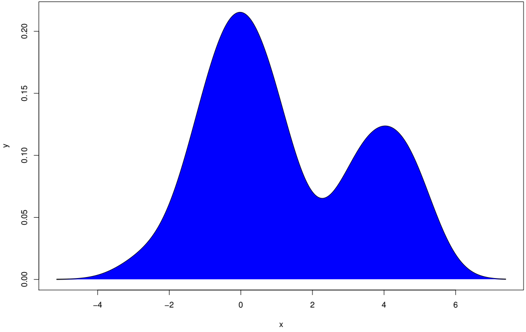

Figure : Example of probability distribution

Bernoulli trial

A Bernoulli distribution is a distribution with two possible outcomes (e.g. heads and tails) and a probability attached to each outcome.

A Bernoulli trial is "an experiment in which s trials are made of an event, with probability p of success in any given trial."

(Weisstein, Eric W. "Bernoulli Trial." http://mathworld.wolfram.com/BernoulliTrial.html)

Binomial distribution

The binomial distribution is "the (...) probability distribution (...) of obtaining exactly n successes out of N Bernoulli trials."

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$$P(n|N) = \frac{N!}{n!(N-n)!} p^n (1-p)^{N-n}$$

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Binomial distribution

$$P(n|N) = \frac{N!}{n!(N-n)!} p^n (1-p)^{N-n}$$

$$\lim_{N\to\infty} p(n) = \frac{1}{\sqrt{2\pi Npq}} e^{-\frac{(n-Np)^2}{2Npq}}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(n-\bar{n})^2}{2\sigma^2}}$$

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i.e. the limiting distribution of the binomial distribution is the normal distribution, with $\sigma^2 \equiv Npq$.

(Weisstein, Eric W. "Binomial distribution."

Normal distribution

Also called Gaussian distribution

Normal distribution

Also called Gaussian distribution, but Gauss did not invent it.

(Davidson & MacKinnon 1999: 130-135)

χ^2-distribution

The sum of squares of r independent normal distributions, is distributed chi-squared with r degrees of freedom:

$$\chi^2(r) \equiv \sum_i^r x_i^2$$

(Weisstein, Eric W. "Chi-squared distribution." http://mathworld.wolfram.com/chi-squareddistribution.html)

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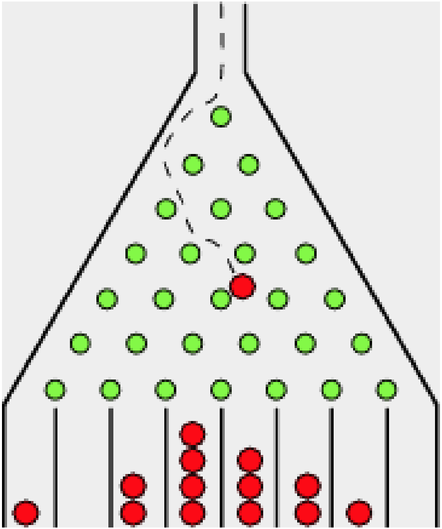
Galton board



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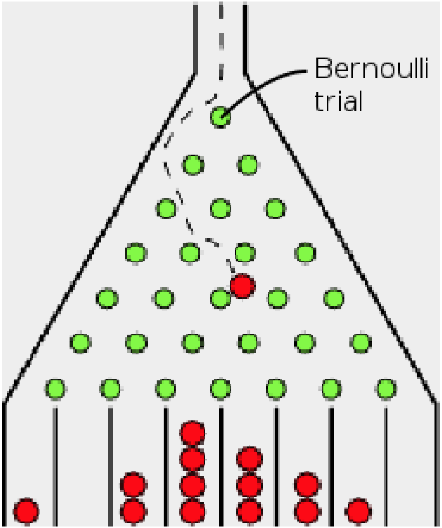

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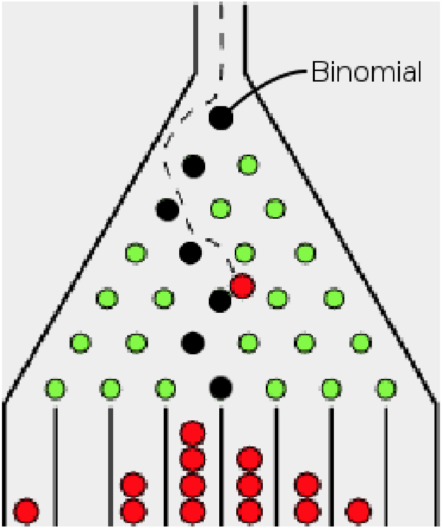

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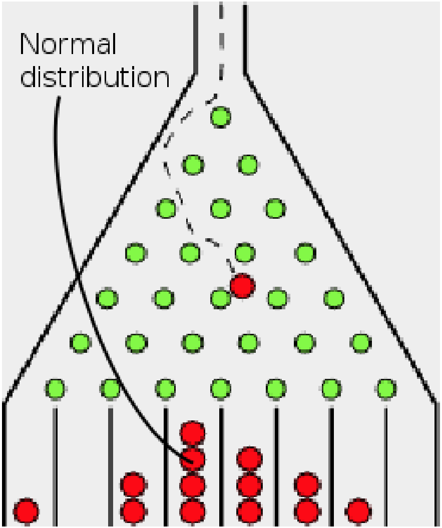

Normal distribution

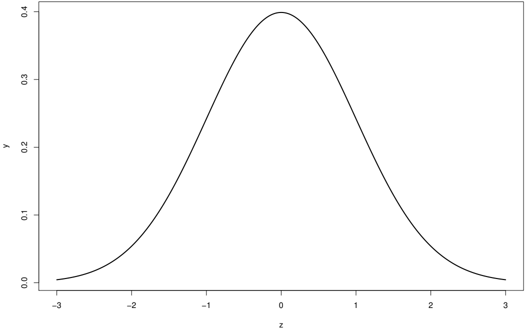

Figure : Normal distribution

Normal distribution

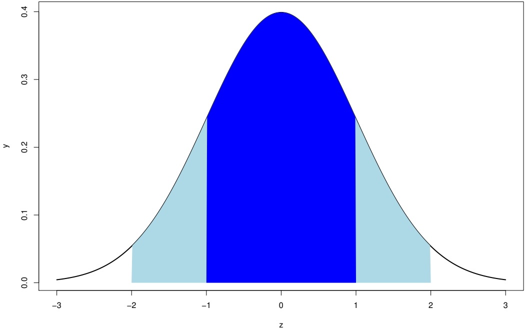

Figure : Normal distribution, with 68% and 95% zones.

Normal distribution

$$f(x|\mu,\sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$$

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Thus entirely determined by the mean μ and the standard deviation σ.

Normal distribution: notation

$$y_i \sim N(\mu, \sigma^2)$$

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Y is normally distributed with mean μ and variance σ^2.

z-scores

If a variable follows a normal distribution, we can standardize it, converting the values of the variable to z-scores.

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Definition: the z-score of a value y_i for variable Y is the number of standard deviations between y_i and the mean of Y, $\bar{y}$.

$$z_i = \frac{y_i - \bar{y}}{s_y} \qquad z_i \sim N(0, 1)$$

Standard normal distribution

The $N(0,1)$ distribution is called the standard normal distribution.

Calculating z-scores for a variable is called standardizing the variable.

Rules of thumb

The surface under the probability distribution between $z = -1$ and $z = 1$ is approximately 69%.

Between $z = -2$ and $z = 2$ is 95%.

z-scores: exercise

"The local police force in Shinbone, Kansas, gives all applicants an entrance exam and accepts only those applicants who score in the top 15% on this test. If the mean score this year is 87 and the standard deviation is 8, would an individual with a score of 110 be accepted?"

(Healey 1996: 128)

t-distribution

When σ is not known, but estimated, we use the t-distribution. When N is sufficiently large, the t-distribution approximates the normal distribution.

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Exercise

"A 25-item scale measuring political conservatism has been administered to a sample of 500 respondents. The scores of eight respondents are listed below."

score	z-score	% below
19		
10		
14		
15		
15		
18		
20		
22		

Exercise

"A scale measuring prejudice has been administered to a large sample of respondents. The distribution of scores is approximately normal with a mean of 31 and a standard deviation of 5. If a score of 40 or more is considered 'highly prejudiced,' what is the probability that a person selected at random will have a score in that range?"

(Healey 1996: 129)

Exercise

"Assume that the distribution of a college entrance exam is normal with a mean of 500 and a standard deviation of 100."

score	z-score	% above
650		
400		
375		
586		
437		
526		
621		

(Healey 1996: 127)

Exercise

Open the `demdev.dta` data file.

Standardize the **polity2** and **laggdppc** variables. Plot histograms and a scatterplot of the relation both before and after standardization. What has changed?

Standardizing variables in SPSS:

```
DESCRIPTIVES VARIABLES = ...  / SAVE.
```